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Infinite Graphs with a Nontrivial Bond Percolation Threshold: Some Sufficient Conditions

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We consider Bernoulli bond percolation on infinite graphs and we identify a class of graphs for which the critical percolation probability is strictly less than 1. The graphs in this class have to fulfill conditions stated in terms of a minimal cut set property and a logarithmic isoperimetric inequality. For the particular case of planar graphs the condition on minimal cut sets can be be replaced by the assumption that the dual of the graph is bounded degree.

KEY WORDS: Percolation; infinite graphs; Peierls argument.

In memory of Mario Lucertini

1. INTRODUCTION

Percolation on the hypercubic d-dimensional lattice $\mathbb{Z}^d$ is a widely investigated subject and its connection with statistical mechanics has been exploited since a long time. However, only in recent years there has been an increasing interest about percolative processes on general graphs. Rigorous results about percolative processes beyond $\mathbb{Z}^d$ were first obtained in the early nineties on regular tree or tree-like graphs.[10, 17, 19] In 1996 Benjamini and Schramm[5] proposed a comprehensive study of percolation on general graphs, with special focus on Cayley graphs, quasi-transitive graphs and

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planar graphs. In this work the authors proposed several conjectures, and in particular they raised the question to establish the class of graphs which exhibit a non trivial critical percolation probabilility p_c. They proved in particular that $p_c < 1$ for graphs with positive Cheeger constant (trees are in this class of graphs) and they conjectured that $p_c < 1$ for a quite large class of quasi-transitive graphs, i.e., Cayley graphs of finitely generated infinite groups which are not a finite extension of $\mathbb{Z}$ (most of the regular lattices fall in this class). This seminal work has been followed by several papers. In particular, the conjecture above about Cayley graphs has been shown to be true in the papers.[2, 18, 21] The study of percolation on quasi-transitive graphs has also been further deepened in refs. 3, 4, and 14. Moreover, the investigation of percolation process (and other probabilistic and/or statistical mechanics processes) on tree-like graphs and, in general, on non-amenable graphs (e.g., graphs with non-zero Cheeger constant) has been continued in several works, see, e.g., refs. 7, 13, 15, 16, 20, 25, and 26.

On the other hand, the question related to the geometric/topological structure that a general graph must possess in order to exhibit a non trivial threshold for bond percolation probability gained importance due to ref. 12, which showed that $p_c(G) < 1$ is equivalent to the existence of a phase transition for several other statistical-physics models on G. E.g., if G is a bounded degree graph then $p_c < 1$ also implies that the Ising model on G exhibits a phase transition. Similar results were also given in refs. 8, 23, and 24.

In the present paper we make some improvement in this direction, by establishing a previously unknown class of infinite graphs with a non trivial bond percolation threshold. Graphs $\mathbb{G}$ in this class have to be bounded degree and to fullfill the following properties:

(i) An isoperimetric inequality, stating that the edge boundary of any connected set in $\mathbb{G}$, rooted in a fixed vertex, has to increase at least as the logarithm of the diameter of the set.

(ii) A minimal cut set property, stating that the cut sets of $\mathbb{G}$ are r-connected for some constant r.

In the special case of planar graphs the assumption that the graph is bounded degree can be relaxed and it is enough to assume the graph just locally finite. Moreover for planar graphs (ii) can be replaced by the assumption that the dual of the graph is bounded degree, i.e., that the faces of the graph have a number of edges uniformly bounded.

As we shall see in Section 5, within this class of planar graphs we identify a class $\mathcal{G}$ which includes all the 1-skeletons of the normal tilings of the plane.

2. DEFINITIONS AND NOTATIONS

Let V be a finite or countable set, then $|V|$ is denoting the cardinality of V. We denote by $\mathrm{P}_2(V)$ the set of all subsets $U \subset V$ such that $|U| = 2$. A *graph* is a pair $G = (V, E)$ with V being a countable set, and $E \subset \mathrm{P}_2(V)$. The elements of V are called *vertices* of G and the elements of E are called *edges* of G. A graph $G = (V, E)$ is *finite* if $|V| < \infty$, and infinite otherwise. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. Then $G \cup G' = (V \cup V', E \cup E')$. If $V' \subseteq V$ and $E' \subseteq E$, then G' is a *subgraph* of G, written as $G' \subseteq G$.

Two vertices x and y of G are *adjacent* if $\{x, y\}$ is an edge of G. The *degree* d_x of a vertex $x \in V$ in G is the number of vertices y adjacent to x. A graph $G = (V, E)$ is *locally finite* if $d_x < +\infty$ for all $x \in V$, and it is *bounded degree, with maximum degree* Δ, if $\max_{x \in V} \{d_x\} \leqslant \Delta < \infty$. A graph $G = (V, E)$ is *connected* if for any pair B, C of subsets of V such that $B \cup C = V$ and $B \cap C = \emptyset$, there is an edge $e \in E$ such that $e \cap B \neq \emptyset$ and $e \cap C \neq \emptyset$. Unless otherwise stated, the graphs considered hereafter are connected.

A *path* in a graph G is a sub-graph $\tau = (V_\tau, E_\tau)$ of G such that

$$V_\tau = \{x_1, x_2, \ldots, x_n\} \; E_\tau = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}\}$$

where all x_i are distinct. The vertices x_1 and x_n are called end-vertices of the path, while the vertices $x_2, \ldots, x_{n-1}$ are called the inner vertices of τ and we say that τ connects (or links) x_1 to x_n, (as well as τ is a path from x_1 to x_n). The length $|\tau|$ of a path $\tau = (V_\tau, E_\tau)$ is the number of its edges, i.e., $|\tau| = |E_\tau|$.

Given a graph $G = (V, E)$ and two distinct vertices $x, y \in V$, we denote by $\mathscr{P}_G^{xy}$ the set of all paths in G connecting x to y. The *distance* $d_G(x, y)$ in G between x and y is the number $d_G(x, y) = \min\{|\tau| : \tau \in \mathscr{P}_G^{xy}\}$. Note that $d_G(x, y) = 1$ if and only if $\{x, y\} \in E$. Given two edges e and e' of G, we define $d_G(e, e') = \min\{d_G(x, y) : x \in e, y \in e'\}$. If $S, R \subset V$ then $d_G(S, R) = \min\{d_G(x, y) : x \in S, y \in R\}$. If $F, H \subset E$ then $d_G(F, H) = \min\{d_G(e, e') : e \in F, e' \in H\}$.

Let $G = (V, E)$ be an *infinite* graph. A *ray* $\rho = (V_\rho, E_\rho)$ in G is an *infinite* sub-graph of G such that

$$V_\rho = \{x_1, x_2, \ldots, x_n, \ldots\} \qquad E_\rho = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \ldots\}$$

where all x_i are distinct. The vertex x_1 is called the *starting vertex* of the ray and we say that ρ *starts at* x_1. We denote by $\mathscr{R}_G^x$ the set of all rays in G starting at x. A *geodesic ray* in G is a ray ρ such that, if $V_\rho = \{x_0, x_1, x_2, \ldots, x_n, \ldots\}$, then $d_G(x_0, x_n) = n$ for all $n \in \mathbb{N}$.

Let ρ and ρ' be two geodesic rays starting at x with vertex sets $V_\rho = \{x = x_0, x_1, x_2, \ldots, x_n, \ldots\}$ and $V_{\rho'} = \{x = y_0, y_1, y_2, \ldots, y_n, \ldots\}$ respectively. If V_ρ and $V_{\rho'}$ are such that $d_G(x_n, y_m) = n+m$ for any $\{n, m\} \in \mathbb{N}$, then the union $\delta = \rho \cup \rho'$ is called a a *geodesic diameter* (or *bi-infinite geodesic*) in G.

Given $G = (V, E)$ connected and $R \subset V$, let $E|_R = \{\{x, y\} \in E : x \in R, y \in R\}$ and define the graph $G|_R = (R, E|_R)$. Note that $G|_R$ is a sub-graph of G. We call $G|_R$ *the restriction of* G *to* R. We say that $R \subset V$ is *connected* if $G|_R$ is connected. For any non empty $R \subset V$, we further denote by $\partial_e R$ the (edges) boundary of R defined by

$$\partial_e R = \{e \in E : |e \cap R| = 1\} \tag{2.1}$$

If $R \subset V$ we denote

$$\mathrm{diam}(R) = \sup_{\{x, y\} \subset R} d_G(x, y)$$

If $x \in V$ we denote $B(x, r_0)$ the ball of radius r_0 and center at x, namely $B(x, r_0) = \{y \in V : d_G(x, y) \leqslant r_0\}$.

A graph is *planar* if it can be drawn in a plane without edges crossing.

It is possible for a planar graph $G = (V, E)$ to define the *dual* graph $G^* = (V^*, E^*)$ in the following way: let a *face* of a planar graph be each simply connected region of the plane delimited by the edges of G. Now take as set of the vertices V^* of G^* the set of the faces of G; two faces in V^* forms an edge $e^* \in E^*$ if they have an edge $e \in E$ in common. With this definitions, each edge $e^* \in E^*$ cuts exactly one edges $e \in E$ and viceversa, i.e., there is a one-to-one correspondence between edges in E and edges in E^*. Given $\gamma \subset E$ we denote by $\gamma^* \subset E^*$ the unique set in E^* associated to γ by such one-to-one correspondence. Note also that the possibility $e^* = \{z, z\}$ with $z \in V^*$ is not excluded; it occurs whenever $e = \{x, y\}$ with $d_x = 1$.

3. CUT SETS, MINIMAL CUT SETS, AND PEIERLS CONTOURS ON INFINITE GRAPHS

In order to apply a Peierls type argument for percolation on general graphs we need to introduce the concept of cut sets and minimal cut sets of a graph. Such cut sets may be regarded as the generalization of the concept of Peierls contours used in the d-dimensional Ising model.

Hereafter $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ will denote a locally finite infinite connected graph.

A set $\gamma \subset \mathbb{E}$ is called a *cut set* if the graph $\mathbb{G}_\gamma = (\mathbb{V}, \mathbb{E} - \gamma)$ is disconnected. A cut set $\gamma \subset \mathbb{E}$ is called a *minimal cut set* if for all $e \in \gamma$ the set $\gamma - e$ is not a cut set. A *finite* minimal cut set $\gamma \subset \mathbb{E}$ in $\mathbb{G}$ have the following property.

Proposition 1. Let γ be a finite minimal cut set in $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, then it may occur only one of the following two possibilities:

(i) $\mathbb{G}_\gamma$ has no finite connected components,

(ii) $\mathbb{G}_\gamma$ has one and only one finite connected component $A_\gamma = (I_\gamma, E_\gamma)$ with $E_\gamma = \mathbb{E}|_{I_\gamma}$) and $\gamma = \partial_e I_\gamma$.

Proof. Let us denote by $\mathbb{G}_\gamma^{\text{ext}} = (\mathbb{V}_\gamma^{\text{ext}}, \mathbb{E}_\gamma^{\text{ext}})$ the union of all infinite components of $\mathbb{G}_\gamma$. Let now $\mathbb{G}_\gamma^{\text{int}}$ be the graph with vertex set $I_\gamma = \mathbb{V} - \mathbb{V}_\gamma^{\text{ext}}$ and with edge set $E_\gamma = \mathbb{E} - (\gamma \cup \mathbb{E}_\gamma^{\text{ext}})$. If $I_\gamma = \emptyset$ (and consequently $E_\gamma = \emptyset$) then $\mathbb{G}_\gamma$ has no finite connected components and we are in the case (i). Assume that $I_\gamma \neq \emptyset$. In this case $\mathbb{G}_\gamma^{\text{int}}$ must be connected since by definition of minimal cut set, for any edge $e \in \gamma$, the graph $(\mathbb{V}_\gamma^{\text{ext}} \cup I_\gamma, \mathbb{E}_\gamma^{\text{ext}} \cup E_\gamma \cup e) = (\mathbb{V}, \mathbb{E} - (\gamma - e))$ is connected and this would not be possible if $\mathbb{G}_\gamma^{\text{int}}$ is done by more than one connected components. Finally, the identity $E_\gamma = \mathbb{E}|_{I_\gamma}$ follows immediately from the fact that $E_\gamma \cup \mathbb{E}_\gamma^{\text{ext}} \cup \gamma = \mathbb{E}$. ▮

If case (ii) occurs we say that γ is *a Peierls contour*. In this case the set $I_\gamma \subset V$ is called *the interior of* γ and the graph A_γ, uniquely determined by γ, is called the *animal associated to the Peierls contour* γ. We say that a Peierls contour γ *surrounds* $x \in V$ if $x \in I_\gamma$. Peierls contours in $\mathbb{G}$ have the following property.

Proposition 2. Let γ be a Peierls contour in $\mathbb{G}$ surrounding x, then for any ray $\rho = (V_\rho, E_\rho)$ in $\mathbb{G}$ starting at x we have that $E_\rho \cap \gamma \neq \emptyset$.

Proof. Suppose by absurd that $E_\rho \cap \gamma = \emptyset$. Then $E_\rho \subset E_\rho^1 \cup E_\rho^2$ with $E_\rho^1 \subset E_\gamma$ and $E_\rho^2 \subset \tilde{\mathbb{E}}_\gamma^{\text{ext}}$ where $\tilde{G}_\gamma^{\text{ext}} = (\tilde{\mathbb{V}}_\gamma^{\text{ext}}, \tilde{\mathbb{E}}_\gamma^{\text{ext}})$ is some (infinite) connected component of $\mathbb{G}_\gamma^{\text{ext}}$. The case $E_\rho^2 = \emptyset$ would imply that $E_\rho \subset E_\gamma$ which is impossible since E_ρ is infinite and E_γ is finite. The case $E_\rho^1 = \emptyset$ is impossible since no edge in $\mathbb{E}_\gamma^{\text{ext}}$ has x as one of its end-points. Finally the last case $E_\rho^1 \neq \emptyset$ and $E_\rho^2 \neq \emptyset$ is impossible since otherwise $A_\gamma \cup \tilde{\mathbb{G}}_\gamma^{\text{ext}}$ would be connected which contradicts Proposition 1. ▮

We denote by $\Gamma_{\mathbb{G}}$ the set of all Peierls contours in $\mathbb{G}$ and by $\Gamma_{\mathbb{G}}^x$ the set of Peierls contours in $\mathbb{G}$ surrounding x. There is clearly a one-to-one correspondence between Peierls contours and connected finite sets in $\mathbb{G}$. Namely, if $\Theta_{\mathbb{G}}$ denotes the set of all connected and finite subsets of $\mathbb{V}$, then Proposition 1 implies immediately that the map $\gamma \mapsto I_\gamma$ is bijection from $\Gamma_{\mathbb{G}}$

into $\Theta_{\mathbb{G}}$. As a matter of fact, if $R \subset \mathbb{V}$ is connected and finite, then $\partial_e R$ is a Peierls contour with interior R.

4. BERNOULLI PERCOLATION ON $\mathbb{G}$ AND THE PEIERLS ARGUMENT

A percolation process on $\mathbb{G}$ is defined as follows. Suppose that each edge in $\mathbb{G}$ can have two possible states. Namely an edge $e \in \mathbb{E}$ can be either *open* or *closed*. A *configuration* of the system is a function $\omega: \mathbb{E} \to \{0, 1\}$ assigning to each edge $e \in \mathbb{E}$ either the value 1 (open edge) or 0 (closed edge). Let Ω be the set of all configurations of the system.

In the Bernoulli bond percolation, the edges of $\mathbb{G}$ are *open* with independent probability p and closed with independent probability $1-p$. The product measure of the configurations of edges is denoted by P_p.

Let $C(x)$ be the open cluster of the vertex x, namely $C(x)$ is the connected set of open edges (possibly empty) containing x. Then define

$$\theta_x(p) = P_p\{C(x) \text{ is infinite}\} \tag{4.1}$$

and the *critical probability of percolation* on $\mathbb{G}$ as

$$p_c = \sup\{p\colon \theta_x(p) = 0\} \tag{4.2}$$

We recall that by FKG inequality p_c is independent of the choice of the vertex x (see, e.g., Theorem 2.8 in ref. 9).

Let $\gamma \in \Gamma_{\mathbb{G}}$ be a Peierls contour, we denote by S_γ the subset of $\mathbb{V}$ defined as $S_\gamma = \{x \in \mathbb{V} : x \in e \text{ for some } e \in \gamma\}$ and call it *the support* of γ.

Let $W \subset \mathbb{E}$, and define $L(W) = \sup_{W_1 \cup W_2 = W} d_{\mathbb{G}}(W_1, W_2)$. A set $W \in \mathbb{E}$ is r-close if $L(W) \leqslant r$.

A graph $\mathbb{G}$ has the *minimal cut set property* if it exists $r_0 < \infty$ such that any minimal cut set γ is r_0-close.

When $\mathbb{G}$ is planar, if $\gamma \in \Gamma_{\mathbb{G}}$ is a Peierls contour then its dual $\gamma^* \subset \mathbb{E}^*$ is a *simple cycle* in $\mathbb{G}^*$, i.e., a path in $\mathbb{G}^*$ that starts in a vertex of $\mathbb{V}^*$ and ends in the same vertex and doesn't repeat a vertex except for the first and last.

It is now easy to state the Peierls argument for our percolation process on $\mathbb{G}$. Consider

$$1 - \theta_x(p) = P_p\{|C(x)| < \infty\}$$

Now observe that for any configuration ω such that $|C(x)| < \infty$ it exists at least a Peierls contour γ of closed edges surrounding x. The probability of such Peierls contour γ is simply $(1-p)^{|\gamma|}$. Hence

$$1 - \theta_x(p) \leqslant \sum_{\gamma \in \Gamma^x_{\mathbb{G}}} (1-p)^{|\gamma|} = \sum_{n \geqslant 1} (1-p)^n \sum_{\substack{\gamma \in \Gamma^x_{\mathbb{G}} \\ |\gamma| = n}} 1 \tag{4.3}$$

Therefore in order to prove that $p_c < 1$ one has simply to show that it exists a constant K such that, for all $n \in \mathbb{N}$

$$\sum_{\substack{\gamma \in \Gamma^x_{\mathbb{G}} \\ |\gamma| = n}} 1 \leqslant K^n \tag{4.4}$$

It is important to remark that the techniques developed in refs. 7 and 22 could be used also to further prove (easily) that the function $\theta_{x_0}(p)$ is actually analytic in the neighborhood of $p = 1$ and so are the finite connectivity functions.

5. RESULTS

We will prove in this section three theorems. The first one is an easy consequence of the results in Section 4 of ref. 2 and identify a class of bounded degree graphs for which $p_c < 1$. Note that for our purposes the conditions listed in Proposition 8 in ref. 2 may be slightly weakened. The second theorem identifies a different class of bounded degree graphs for which $p_c < 1$; in particular the request of existence of a bi-infinite geodesic is replaced by a very weak isoperimetric inequality condition.

In the third theorem we show that for planar graphs again $p_c < 1$ for a very large class of graphs, including graphs locally finited but not bounded degree.

Theorem 1. Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be bounded degree, with maximum degree Δ, such that:

(i) $\mathbb{G}$ has the *minimal cut set property*.

(ii) There exists a bi-infinite geodesic in $\mathbb{G}$.

Then the critical probability p_c on $\mathbb{G}$, defined in (4.2), is strictly less than 1.

Proof. Take x on the bi-infinite geodesic. We want to bound

$$\sum_{\substack{\gamma \in \Gamma^x_{\mathbb{G}} \\ |\gamma| = n}} 1 \tag{5.1}$$

First note that, by (i), there exists r_0 such that every Peierls contour $\gamma \in \Gamma_{\mathbb{G}}$ is r_0-close. Therefore, a Peierls contour γ with cardinality n has every pair of edges at most $r_0 n$ far apart.

Consider now the geodesic diameter $\delta = \rho \cup \rho'$ in $\mathbb{G}$ from x, then, by Proposition 2, $E_\rho \cap \gamma \neq \emptyset$ and $E_{\rho'} \cap \gamma \neq \emptyset$. Let $e_x(\gamma)$ (let $e'_x(\gamma)$) be the first edge, in the natural ray order, of E_ρ (of $E_{\rho'}$) which belongs to γ and define

$$\delta_n(x) = \{e \in E_\rho : \exists \gamma \in \Gamma^x_{\mathbb{G}} \text{ such that } |\gamma| = n \text{ and } e = e_x(\gamma)\}$$

Thus

$$\sum_{\substack{\gamma \in \Gamma^x_{\mathbb{G}} \\ |\gamma| = n}} 1 = \sum_{e \in \delta_n(x)} \sum_{\substack{\gamma \in \Gamma^x_{\mathbb{G}} : |\gamma| = n \\ e_x(\gamma) = e}} 1 \leqslant |\delta_n(x)| \sup_{e \in \mathbb{E}} \sum_{\substack{\gamma \in \Gamma_{\mathbb{G}}\, |\gamma| = n \\ e \in \gamma}} 1$$

Now, since $\rho \cup \rho'$ is bi-infinite geodesic, $|\delta_n(x)| \leqslant \max_\gamma d_{\mathbb{G}}(e_x(\gamma), e'_x(\gamma))$. Since $\{e_x(\gamma), e'_x(\gamma)\} \subset \gamma$ and γ is r_0 close, this means that $d_{\mathbb{G}}(e_x(\gamma), e'_x(\gamma) \leqslant r_0 n$ and we obtain the bound

$$|\delta_n(x)| \leqslant r_0 n$$

To bound the number of Peierls contour of cardinality n containing a fixed edge e, we proceed as follows. Let now $\mathbb{G}_{r_0}$ be the graph with vertex set $\mathbb{V}$ and edge set $\mathbb{E}_{r_0} = \{\{x, y\} : d_{\mathbb{G}}(x, y) \leqslant r_0\}$. $\mathbb{G}_{r_0}$ is a connected bounded degree graph with maximum degree $\Delta_{\rho_0} = \max_{x \in \mathbb{V}} |B(x, r_0)| \leqslant \Delta^{r_0+1}$. Consider now the support S_γ of γ. Since γ is r_0-close, then the restriction $S_\gamma|_{\mathbb{G}_{r_0}}$ is connected in $\mathbb{G}_{r_0}$. Moreover $1 \leqslant |S_\gamma| \leqslant 2n$. Therefore

$$\sup_{e \in \mathbb{E}} \sum_{\substack{\gamma \in \Gamma_{\mathbb{G}}\, |\gamma| = n \\ e \in \gamma}} 1 \leqslant \sup_{x \in \mathbb{V}} \sum_{k=1}^{2n} \sum_{\substack{W \subset \mathbb{V} : |W| = k \\ W|_{\mathbb{G}_{r_0}} \text{ connected, } x \in W}} \leqslant \sum_{k=1}^{2n} \Delta^{2k}_{\rho_0},$$

i.e., finally

$$\sum_{\gamma \in \Gamma^x_{\mathbb{G}} : |\gamma| = n} 1 \leqslant 2rn \sum_{k=1}^{2n} \Delta^{2k}_{\rho_0} \leqslant 2rn \sum_{k=1}^{2n} [\Delta^{r_0+1}]^{2k} \leqslant C^n$$

for some constant C depending only on Δ and r_0. ∎

Theorem 2. Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be bounded degree, with maximum degree Δ, such that:

(i) $\mathbb{G}$ has the *minimal cut set property*

(ii) It is possible to find a uniform constant C and a vertex $x_0 \in \mathbb{V}$ such that, for all finite connected $W \subset \mathbb{V}$ with $x_0 \in W$, the inequality

$$|\partial_e W| \geqslant \frac{1}{C} \ln \operatorname{diam}(W) \tag{5.2}$$

is verified.

Then the critical probability p_c on $\mathbb{G}$, defined in (4.2), is strictly less than 1.

Proof. To bound the factor (5.1) we now proceed as follows. Since $\mathbb{G}$ is connected and locally finite, for any $x \in \mathbb{V}$ hence in particular for x_0, there exists a geodesic ray ρ starting at x_0. Then, since $\gamma \in \Gamma_{\mathbb{G}}^{x_0}$, we have, by Proposition 2, that $E_\rho \cap \gamma \neq \varnothing$. Let $e_{x_0}(\gamma)$ be the first edge (in the natural order of the ray) in E_ρ which belongs to γ and define

$$r_n(x_0) = \{e \in E_\rho : \exists \gamma \in \Gamma_{\mathbb{G}}^{x_0} \text{ such that } |\gamma| = n \text{ and } e = e_{x_0}(\gamma)\} \tag{5.3}$$

Hence

$$\sum_{\substack{\gamma \in \Gamma_{\mathbb{G}}^{x_0} \\ |\gamma| = n}} 1 = \sum_{e \in r_n(x_0)} \sum_{\substack{\gamma \in \Gamma_{\mathbb{G}}^{x_0} x : |\gamma| = n \\ e_x(\gamma) = e}} 1 \leqslant |r_n(x)| \sup_{e \in \mathbb{E}} \sum_{\substack{\gamma \in \Gamma_{\mathbb{G}}\, |\gamma| = n \\ e \in \gamma}} 1$$

Now observe that

$$|r_n(x)| \leqslant \sup_{\gamma \in \Gamma_{\mathbb{G}}^{x_0} : |\gamma| = n} \operatorname{diam}(I_\gamma) + 1$$

and by hypothesis (ii) of the theorem, i.e., inequality (5.2) we get

$$|\rho_n(x)| \leqslant C^n + 1$$

The factor $\sup_{e \in \mathbb{E}} \sum_{\substack{\gamma \in \Gamma_{\mathbb{G}}\, |\gamma| = n \\ e \in \gamma}} 1$ can now be bounded exactly as in Theorem 1. ▮

Theorem 3. Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a planar locally finite connected infinite graph with the following properties

(i) Its dual graph $\mathbb{G}^*$ is bounded degree with maximum degree Δ^*.

(ii) It is possible to find a uniform constant C and a vertex $x_0 \in \mathbb{V}$ such that, for all finite connected $W \subset \mathbb{V}$ with $x_0 \in W$, the inequality

$$|\partial_e W| \geqslant \frac{1}{C} \ln \operatorname{diam}(W) \tag{5.4}$$

is verified.

Then the critical probability p_c on $\mathbb{G}$, defined in (4.2), is strictly less than 1.

Proof. Proceeding as in Theorem 2, consider the geodesic ray ρ from x_0. Then, since $\gamma \in \Gamma^{x_0}_{\mathbb{G}}$, we have that $E_\rho \cap \gamma \neq \emptyset$. Let $e_{x_0}(\gamma)$ be the first edge (in the natural order of the ray) in E_ρ which belongs to γ and define the edge set $r_n(x_0)$ as in (5.3).

Then

$$\sum_{\substack{\gamma \in \gamma \in \Gamma^{x_0}_{\mathbb{G}} \\ |\gamma| = n}} 1 = \sum_{e \in r_n(x_0)} \sum_{\substack{\gamma \in \Gamma^{x_0}_{\mathbb{G}} : |\gamma| = n \\ e_{x_0}(\gamma) = e}} 1 \leqslant |r_n(x_0)| \sup_{e \in \mathbb{E}} \sum_{\substack{\gamma \in \Gamma_{\mathbb{G}}\ |\gamma| = n \\ e \in \gamma}} 1$$

where, as before,

$$|\rho_n(x_0)| \leqslant C^n + 1$$

We now use the one-to-one correspondence between Peierls contours of cardinality n in $\mathbb{G}$ and simple cycles of length n in $\mathbb{G}^*$, and we write

$$\sup_{e \in \mathbb{E}} \sum_{\substack{\gamma \in \Gamma_{\mathbb{G}}\ |\gamma| = n \\ e \in \gamma}} 1 = \sup_{e^* \in \mathbb{E}^*} \sum_{\substack{\gamma^* \in \Gamma^*_{\mathbb{G}}\ |\gamma^*| = n \\ e^* \in \gamma^*}} 1 \tag{5.5}$$

where $\Gamma^*_{\mathbb{G}}$ is the set of all simple cycles in $\mathbb{G}^*$. Using now hypothesis (i) of the theorem, it is now trivial to bound the l.h.s. of (5.5). We get

$$\sup_{e^* \in \mathbb{E}^*} \sum_{\substack{\gamma^* \in \Gamma^*_{\mathbb{G}}\ |\gamma| = n \\ e^* \in \gamma^*}} 1 \leqslant [\Delta^*]^n \tag{5.6}$$

Hence we have again proved that

$$\sum_{\gamma \in \Gamma_{x_0} : |\gamma| = n} 1 \leqslant (C^n + 1)[\Delta^*]^n \leqslant K^n$$

for some constant K. ▮

6. REMARKS AND EXAMPLES

First we note that in general the hypothesis (ii) of Theorem 1, namely the existence of the bi-geodesic, is quite a strong condition on the graph and in many cases it is not easy to be verified. Theorem 2, on the other side, replaces it with a very mild isoperimetric condition. Such condition appears to be optimal. More specifically the growth of the boundary of a connected set with the log of the diameter generalizes a result presented in ref. 9 about percolation of infinite subsets of the (two-dimensional) square lattice.

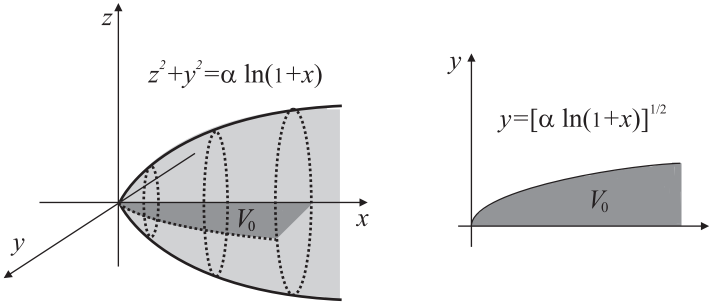

Fig. 1. A sketch of the (infinite) regions delimiting the subset $V \subset \mathbb{Z}^3$ and the subset $V_0 \subset Z^2$.

As an example, let $\mathbb{L}^3 = (\mathbb{Z}^3, \mathbb{E}^3)$ be the usual three-dimensional cubic lattice with vertex set $\mathbb{Z}^3 = \{x = (m_1, m_2, m_3) : m_i \in \mathbb{Z}\}$ and edge set $\mathbb{E}^d$ formed by the nearest neighbor pairs of $\mathbb{Z}^3$. This is a bounded degree graph which has the minimal cut set property. We also think to $\mathbb{L}^3$ as naturally embedded in $\mathbb{R}^3$. Let now V be the subset of $\mathbb{Z}^3$ given by

$$V = \{v = (m_1, m_2, m_3) \in \mathbb{Z}^3 : m_1 \geqslant 0,\ m_2^2 + m_3^2 \leqslant \alpha \ln(1 + m_1)$$

Namely V is the infinite set of vertices in $\mathbb{Z}^d$ "inside" the region of $\mathbb{R}^3$ delimitated by the revolution surface of equation $r^2 = \alpha \ln(1+x)$ where $r^2 = y^2 + z^2$, α is a positive constant and $y \geqslant 0$ (see Fig. 1).

Consider now the graph $G = \mathbb{L}^3|_V$ which is the restriction of $\mathbb{L}^3$ to V. This graph does not admit an infinite bi-geodesic and is not planar, so Theorem 1 and Theorem 3 are not useful. But it is easy to see that G satisfies all hypothesis of Theorem 2 and therefore has a non trivial percolation threshold.

Indeed, the graph G is bounded degree and has the minimal cut set property. It is also immediate to check that G satisfies the isoperimetric condition (4.2). The worst cases are clearly when we choose connected subsets of V of the form

$$W_n = \{v = (m_1, m_2, m_3) \in V : m_1 \leqslant n+1\}$$

where $n \geqslant 1$ in such way that $|W_n| \geqslant 2$.

In this case we have that $|\partial_e W_n| \geqslant \pi\alpha \ln(1+n)$ and $\operatorname{diam} W_n \leqslant 2(n+1)$. Hence

$$\frac{|\partial_e W_n|}{\ln(\operatorname{diam} W_n)} \geqslant \frac{\pi\alpha \ln(1+n)}{\ln 2 + \ln(1+n)} \geqslant \frac{\pi\alpha}{2}$$

It is interesting to notice that two-dimensional slices of G are not expected to have a non trivial percolation threshold, i.e., Theorem 11.55 of ref. 9 is not applicable. E.g., take the bi-dimensional (infinite) subgraph $G|_{V_0}$ of G where

$$V_0 = \{v = (m_1, m_2, m_3) \in \mathbb{Z}^3 : m_1 \geqslant 0, m_3 = 0, 0 \leqslant m_2 \leqslant \sqrt{\alpha \ln(1+m_1)}\}$$

The graph $G|_{V_0}$ can be viewed as a subgraph of $\mathbb{Z}^2$ of the type $G(f)$ of Section 11.5 in ref. 9 where $f(u)$ is the non negative function on $[0, \infty)$ given by $f(u) = \sqrt{\alpha \ln(1+u)}$. We immediately see that this graph is not "fat enough" to to ensure non trivial percolation threshold by Theorem 11.55 of ref. 9, since $f(u)/\ln u \to 0$ as $u \to \infty$. So the graph G could provide an example of sub-graph of $\mathbb{Z}^3$ which percolates while no bi-dimensional subsets of this graph possibly percolates. We also remark that this example can easily generalized to $\mathbb{Z}^d$ and other d-dimensional lattice.

Concering now planar graphs by this Theorem 3 one can establish weather a planar graph has or not a non trivial percolation threshold, without using any particular symmetry property of the graph (e.g., transitivity).

In particular, Theorem 3 can be proved easily to be true for the a class of tilings of the plane, usually referred in the literature as the class of *normal tilings* (see, e.g., ref. 11). A tiling $\mathbb{T}$ of the plane is a countable family $\mathbb{T} = \{T_1, T_2, \dots, \}$ of closed sets (called tiles) which cover the plane without gaps or overlaps. A tiling $\mathbb{T}$ is *normal* if every tile $T \in \mathbb{T}$ is a topological disk and it is uniformly bounded, i.e., there exists two positive numbers $U_{\mathbb{T}}$ and $u_{\mathbb{T}}$ such that T contains a disk of radius $u_{\mathbb{T}}$ and it is contained in a disk of radius $U_{\mathbb{T}}$. For normal tilings, the intersection of any pair of tiles is empty or may consists of a set of isolated points or arcs. The isolated points of a tiling are called vertices of the tiling and the arcs are called edges of the tiling.

If $V_{\mathbb{T}}$ denotes the set of all vertices of the tiling and $E_{\mathbb{T}}$ the set of all edges of the tiling, then the graph $G_{\mathbb{T}} = (V_{\mathbb{T}}, E_{\mathbb{T}})$ is called the *1-skeleton* of $\mathbb{T}$. Note that $G_{\mathbb{T}}$ is locally finite if $\mathbb{T}$ is normal.

Now, the following corollary is an easy consequence of Theorem 3.

Corollary 1. Let $\mathbb{T}$ be a normal tiling. Then the graph $G_{\mathbb{T}} = (V_{\mathbb{T}}, E_{\mathbb{T}})$ has bond percolation threshold less than 1.

Proof. It is easy to see that if $\mathbb{T}$ is a normal tiling then $G_{\mathbb{T}}$ is bounded degree and $G^*_{\mathbb{T}}$ is also bounded degree. To prove that inequality (5.2) is satisfied let us consider a set $T_n \subset \mathbb{T}$ formed by $n = |T_n|$ pairwise connected tiles of $\mathbb{T}$ (two tiles are pairwise connected if they have an edge in common). Denote now by $W_n \subset V_{\mathbb{T}}$ the set of vertices in T_n and by P_n the subset of the plane which is the union of the tiles in T_n. Without loss in generality, we can suppose that P_n is a simply connected closed set of the plane. Then the edge boundary $\partial_e W_n$ of W_n is a Peierls contour of $G^*_{\mathbb{T}}$. By hypothesis the dual graph is bounded degree, i.e., a tile can have at most N vertices, so that $|W_n| \leqslant Nn$. Let now $A(P_n)$ be the area of the set P_n and let $L(P_n)$ be the length of the boundary of P_n. By the assumption that the tiling is normal, we have that $A(P_n) \geqslant n(\pi u_{\mathbb{T}}^2)$ and the boundary of P_n contains at least $L_{(P_n)}/U_{\mathbb{T}}$ edges so that $|\partial_e W_n| \geqslant L(P_n)/U_{\mathbb{T}}$. Moreover, since P_n is some closed and simply connected set in the plane, then it is possible to find a constant C such that $L_{(P_n)} \geqslant C[A(P_n)]^{1/2}$. Therefore we get

$$|\partial W_n| \geqslant L(P_n)/U_{\mathbb{T}} \geqslant \frac{C}{U_{\mathbb{T}}}[A(P_n)]^{1/2} \geqslant \frac{C}{U_{\mathbb{T}}}[n(\pi u_{\mathbb{T}}^2)]^{1/2} \geqslant C\sqrt{\pi}\,\frac{u_{\mathbb{T}}}{U_{\mathbb{T}}}\left[\frac{|W_n|}{N}\right]^{1/2}$$

Since now $|W_n|^{1/2} \geqslant \ln[(\mathrm{diam}(W_n)]$, the inequality (4.2) follows. ∎

It is interesting to observe that the class of tilings satisfying the Proposition 3 includes quasi-periodic tiling of the plane such as the Penrose tiling of Fig. 2.

It is finally important to remark that, as a possible further development of methods and ideas illustrated in this paper, one could try to modify Theorem 3 by strengthening the condition (4.2) (which, as we have tried to show, is a a very mild condition) and at the same time by trying to relax the condition that the dual be bounded degree in order to treat

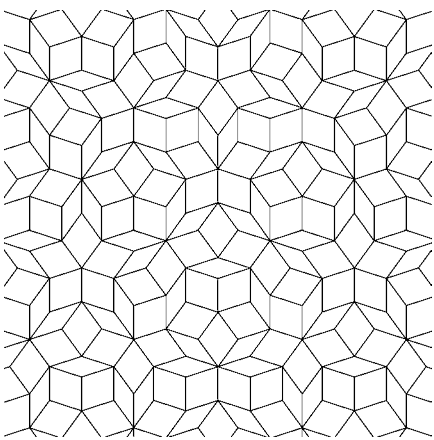

Fig. 2. The Penrose tiling.

proximity random graphs (which in general have a dual not bounded degree) like, e.g., the β-skeletons on point Poisson processes of the plane. These random graphs includes the *Gabriel graph* and the *relative neighborhood graph* which are important for many applications.

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